NOTE ON THE NECESSARY CONDITION THAT TWO LINEAR HOMOGENEOUS DIFFERENTIAL EQUATIONS SHALL HAVE COMMON INTEGRALS.*

By IDA MAY SCHOTTENFELS.

Professor Von Escherich in the *Denkschriften der Wiener Akademie,* Vol. 46, and later Heffter in Crelle's *Journal,* Vol. 116, proved that there exists for linear differential homogeneous equations a theory analogous to that of algebraic equations, confining their researches to the analogues to theorems upon the Highest Common Factor and Lowest Common Multiple.

During the past year Dr. Epsteen and Dr. Pierce revived this subject, and in THE AMERICAN MATHEMATICAL MONTHLY, Vol. X, March, 1903, pp. 63-68, Dr. Epsteen gives the necessary condition for one common integral, while Dr. Pierce gives the sufficient condition for $k \gtreqless 1$ independent common integrals.

This note gives the necessary condition for two common integrals.

The following example illustrates the method.

$$a_0x^4+a_1x^3+a_2x^2+a_3x+a_4=0, \tag{1}$$

$$b_0x^3+b_1x^2+b_2x+b_3=0. \tag{2}$$

To find the necessary condition that equations (1) and (2) have two roots in common. According to the well known dialytic method of elimination of Sylvester we may write

$$\left\{\begin{array}{l} a_0x^5+a_1x^4+a_2x^3+a_3x^2+a_4x \qquad =0, \\ \qquad a_0x^4+a_1x^3+a_2x^2+a_3x+a_4=0, \\ b_0x^5+b_1x^4+b_2x^3+b_3x^2 \qquad\qquad =0, \\ \qquad b_0x^4+b_1x^3+b_2x^2+b_3x \qquad =0, \\ \qquad\qquad b_0x^3+b_1x^2+b_2x+b_3=0. \end{array}\right. \tag{3}$$

Eliminating x^5, x^4, x^3, x^2, from (3) we get

$$\begin{vmatrix} a_0 & a_1 & a_2 & a_3 & a_4x+0 \\ 0 & a_0 & a_1 & a_2 & a_3x+a_4 \\ b_0 & b_1 & b_2 & b_3 & 0x+0 \\ 0 & b_0 & b_1 & b_2 & b_3x+0 \\ 0 & 0 & 0 & b_1 & b_2x+b_3 \end{vmatrix} =0. \tag{4}$$

Since (4) shall hold for at least two values of x, and yet is linear in x, it is an identity in x, and hence

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$$(5)\qquad \begin{vmatrix} a_0 & a_1 & a_2 & a_3 & a_4 \\ 0 & a_0 & a_1 & a_2 & a_3 \\ b_0 & b_1 & b_2 & b_3 & 0 \\ 0 & b_0 & b_1 & b_2 & b_3 \\ 0 & 0 & 0 & b_1 & b_2 \end{vmatrix} \equiv 0, \qquad \begin{vmatrix} a_0 & a_1 & a_2 & a_3 & 0 \\ 0 & a_0 & a_1 & a_2 & a_4 \\ b_0 & b_1 & b_2 & b_3 & 0 \\ 0 & b_0 & b_1 & b_2 & 0 \\ 0 & 0 & 0 & b_1 & b_3 \end{vmatrix} \equiv 0.$$

Hence the required necessary condition that equations (1) and (2) have two roots in common is the vanishing of determinants (5).

The generalizatisn to equations of degree m aud n, that have p roots in common follows readily. The equation of degree m must be multiplied by x^{n-p} while the one of degree n must be multiplied by x^{m-p}.

In exactly the same manner the analog of this method for linear homogeneous differential equations can be developed. The theorem that a linear homogeneous differential equation of the nth order has n and only n linearly independent integrals is employed, and the necessary condition is expressed in linear homogeneous differential operators, and has a form analogous to that of the necessary condition (5).

Given the following two linear homogeneous differential equations:

$$\text{(I)}\qquad \alpha_0(x)y^{\text{iv}}+\alpha_1(x)y'''+\alpha_2(x)y''+\alpha_3(x)y'+\alpha_4(x)y=0,$$

$$\text{(II)}\qquad \beta_0(x)y'''+\beta_1(x)y''+\beta_2(x)y'+\beta_3(x)y=0.$$

If these two equations I and II have at least one common integral, we can by differentiation write,

$$\text{(III)}\begin{cases} \alpha_0 y^{\text{v}}+(\alpha_0'+\alpha_1)y^{\text{iv}}+(\alpha_1'+\alpha_2)y'''+(\alpha_2'+\alpha_3)y''+(\alpha_3'+\alpha_4)y'+\alpha_4'.y\equiv 0 \\ \qquad \alpha_0 y^{\text{iv}}+\alpha_1 y'''+\alpha_2 y''+\alpha_3 y'+\alpha_4 .y\equiv 0 \\ \beta_0 y^{\text{v}}+(2\beta_0'+\beta_1)y^{\text{iv}}+(\beta_0''+2\beta_1'+\beta_2)y'''+(\beta_1''+2\beta_2'+\beta_3)y'' \\ \qquad\qquad +(\beta_2''+2\beta_3')y'+\beta_3''.y\equiv 0 \\ \qquad \beta_0 y^{\text{iv}}+(\beta_0'+\beta_1)y'''+(\beta_1'+\beta_2)y''+(\beta_2'+\beta_3)y'+\beta_3'.y\equiv 0 \\ \qquad \beta_0 y'''+\beta_1 y''+\beta_2 y'+\beta_3 .y\equiv 0. \end{cases}$$

Eliminating y^{v}, y^{iv}, y''', y'' from (III) we get

$$\text{(IV)}\qquad \begin{vmatrix} \alpha_0 & \alpha_0'+\alpha_1 & \alpha_1'+\alpha_2 & \alpha_2'+\alpha_3 & (\alpha_3'+\alpha_4)y'+\alpha_4'.y \\ 0 & \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 y'+\alpha_4 .y \\ \beta_0 & 2\beta_0'+\beta_1 & \beta_0''+2\beta_1'+\beta_2 & \beta_1''+2\beta_2'+\beta_3 & (\beta_2''+2\beta_3')y'+\beta_3''.y \\ 0 & \beta_0 & \beta_0'+\beta_1 & \beta_1'+\beta_2 & (\beta_2'+\beta_3)y'+\beta_3'.y \\ 0 & 0 & \beta_0 & \beta_2 & \beta_2 y'+\beta_3 y \end{vmatrix} \equiv 0$$

which may be written $\lambda y'+\mu y=0$.

If there are two independent common integrals, IV must hold for two linearly independent functions y (one not a constant times the other), whence $\lambda=0$, $\mu=0$, or

$$(\mathrm{V}_1)\quad \begin{vmatrix} \alpha_0 & \alpha_0'+\alpha_1 & \alpha_1'+\alpha_2 & \alpha_2'+\alpha_3 & \alpha_3'+\alpha_4 \\ 0 & \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_0 & 2\beta_0'+\beta_1 & \beta_0''+2\beta_1'+\beta_2 & \beta_1''+2\beta_2'+\beta_3 & \beta_2''+2\beta_3' \\ 0 & \beta_0 & \beta_0'+\beta_1 & \beta_1'+\beta_2 & \beta_2'+\beta_3 \\ 0 & 0 & \beta_0 & \beta_1 & \beta_2 \end{vmatrix} \equiv 0,$$

$$(\mathrm{V}_2)\quad \begin{vmatrix} \alpha_0 & \alpha_0'+\alpha_1 & \alpha_1'+\alpha_2 & \alpha_2'+\alpha_3 & \alpha_4' \\ 0 & \alpha_0 & \alpha_1 & \alpha_2 & \alpha_4 \\ \beta_0 & 2\beta_0'+\beta_1 & \beta_0''+2\beta_1'+\beta_2 & \beta_1''+2\beta_2'+\beta_3 & \beta_3'' \\ 0 & \beta_0 & \beta_0'+\beta_1 & \beta_1'+\beta_2 & \beta_3' \\ 0 & 0 & \beta_0 & \beta_1 & \beta_3 \end{vmatrix} \equiv 0.$$

Hence V_1 and V_2 furnish the required necessary condition.

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LINEAR COVARIANTS OF THE BINARY QUADRATIC AND CUBIC.

By L. C. WALKER, Professor of Mathematics, Colorado School of Mines, Golden, Col.

The definition of *weight* is that every coefficient is of weight w measured by its suffix, and that every product of coefficients is of weight measured by the sum of the suffixes of its various factors.

A semi-covariant of the two quantics is a function of the two sets of coefficients, which is homogeneous in each set separately, and *isobaric* (equal weight) on the whole, though not necessarily in the sets separately.

The practice of speaking of a covariant whose dimensions are partial degrees i_1, i_2 in the two sets of coefficients and ω in the variables has of late become almost universal.*

The degrees of quantics in the variables are generally† spoken of as their orders p_1, p_2.

The order ω, the partial degrees i_1, i_2 in the coefficients of the binary quadratic and cubic

$$(a_0,\ a_1,\ a_2)(x,\ y)^2,\quad (b_0,\ b_1,\ b_2,\ b_3)(x,\ y)^3,$$

and the weight w of the semi-invariant which is the leading coefficient C_0 in the linear covariant, are connected by the relation $i_1p_1+i_2p_2-\omega=2w$.

Here $p_1=2$, $p_2=3$, $\omega=1$, whence $2i_1+3i_2-1=2w$. More generally, if m be any positive integer and n any positive odd integer, we have, from the conditions of linear covariancy, $2mi_1+3ni_2-1=2w$. Thus the binary quadratic and cubic have an indefinite number of linear covariants.

*Elliott's *Algebra of Quantics*. †*Ibid.*